APPROXIMATE SOLUTIONS OF THE HEAT CONDUCTION EQUATION FOR A TWO-LAYER PLATE WITH SUBLIMATION AT THE OUTER SURFACE

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The problem of the heat propagation and sublimation of the outer surface for a plate of finite thickness is considered in the case of an arbitrary dependence of external heat flow on time. The solutions obtained make it possible to determine the total sublimation time or the thickness of the sublimed layer, the temperature distribution in the first layer and the mean temperature of the second layer as a function of time.

In [1] we examined the problem of the heating and sublimation of a semiinfinite rod for an arbitrary dependence of the heat flow on time. Now let the length of the rod be short enough so that the effect of the conditions at the upper end, not exposed to the action of the external heat flow, cannot be neglected. We assume that the rod in question is in close contact with another rod made of a material with different thermophysical characteristics. The sides of both rods are thermally insulated, and, moreover, the second rod also has thermal insulation on the end face turned toward free space. Thus the problem of heat propagation in the system reduces to solving the one-dimensional heat conduction equation. The problem is equivalent to the problem of heating (and sublimation) of a two-layer plate infinite in two directions and exposed to the action of a time-varying heat flow distributed over one of its surfaces (Fig. 1).

Then for the two layers we have the heat conduction equations

$$a_i \frac{\partial^2 T_i}{\partial x_1^2} - \frac{\partial T_i}{\partial t} = 0, \quad i = 1, 2, \ldots. \tag{1}$$

For i = 1 we have $\xi \leq x_1 \leq \delta_1$, and similarly for i = = 2, $\delta_1 \leq x_1 \leq \delta_1 + \delta_2$.

The boundary conditions are

at $x_1 = \xi$

$$T_1(\xi, t) = T_r = \text{const},$$
$$q(t) + \lambda_1 \frac{\partial T_1}{\partial x_1} - \gamma_1 E \dot{\xi} = 0, \tag{2}$$

and at $x_1 = \delta_1$

$$T_1(\delta_1, t) = T_2(\delta_1, t), \; \lambda_1 \frac{\partial T_1}{\partial x_1} = \lambda_2 \frac{\partial T_2}{\partial x_1}.$$

The boundary condition at $x_1 = \delta_1 + \delta_2$ will be considered below. The initial conditions are determined from the solution of the problem for a semiinfinite rod [1]. Thus, we have the function $g(x_1)$ and the constants $\xi(0)$ and $\dot{\xi}(0)$:

$$\text{at } 0 \leqslant x_1 \leqslant \delta_1 \quad T_1(x_1, 0) = g(x_1),$$
$$\dot{\xi}(0) = \dot{\xi}_0, \quad T_2(x_1, 0) = T_0 = \text{const}. \tag{3}$$

In the general case there will also be some temperature distribution other than that indicated in (3) in the second rod. However, we will assume that at the initial instant the temperature of that rod is constant. This corresponds to assuming the presence of a heat-assimilating layer [1]. Then zero time will coincide with the moment at which the thermal front reaches the boundary between the two layers. Furthermore, it is assumed that sublimation on the side of the rod exposed to the heat flow begins before the thermal front reaches the layer boundary.

It follows that the function $g(x_1)$ must satisfy the conditions

$$g(0) = T_r, \quad g(\delta_1) = T_0.$$

A similar problem for a constant heat flux was considered in [2], where it was assumed that a solid homogeneous rod is subjected to heating and melting at one end. The temperature profiles in the solid and liquid phases are given in the form of quadratic parabolas with respect to the coordinate x_1. It should be noted that if these temperature profiles are introduced in the same way as in [2], erroneous results may be obtained. In fact, under the above-mentioned conditions the requirements of the problem are satisfied by the segments of parabolas shown in Fig. 2. Obviously, the picture presented in Fig. 2 is physically quite unreal, but in [2] no limitations are imposed on the functions $T_i(x_1, t)$ that might exclude the possibility of such a situation.

We shall assume that the thermal conductivity of the material of the second rod considerably exceeds that of the first. Then we may approximately consider that the temperature of the second rod does not depend on the coordinate x_1 and varies only with time. In this case the boundary conditions at the junction of the rods take the form:

$$\text{at } x_1 = \delta_1$$
$$T_1(\delta_1, t) = T_2(t), \; \lambda_1 \frac{\partial T_1}{\partial x_1} = c_2 \gamma_2 \delta_2 \frac{dT_2}{dt}. \tag{4}$$

By introducing the new coordinate $x = x_1 - \xi$, tied to the moving sublimation front, we can write the heat conduction equation for the first rod in the following form:

$$a_1 \frac{\partial^2 T_1}{\partial x^2} + \dot{\xi} \frac{\partial T_1}{\partial x} - \frac{\partial T_1}{\partial t} = 0. \tag{5}$$

The boundary conditions are

$$\text{at } x = 0$$

$$T_1(0,\ t)=T_r,\quad q(t)+\lambda_1\frac{\partial T_1}{\partial x}-\gamma_1 E\dot{\xi}=0;$$

$$\text{at } x=\delta_1-\xi$$

$$T_1(\delta_1-\xi,\ t)=T_2(t),\ -\lambda_1\frac{\partial T_1}{\partial x}=c_2\gamma_2\delta_2\frac{dT_2}{dt}. \quad (6)$$

The initial conditions do not change. We require the satisfaction of Eq. (5) in the mean for the layer $\delta_1-\xi$, for which purpose we integrate it with respect to x from $x=0$ to $x=\delta_1-\xi$. Using boundary conditions (6), we obtain the heat balance equation for a two-layer wall:

$$n\delta_1\frac{dT_2}{dt}+\frac{J_1}{c_1}\dot{\xi}+\frac{d\theta}{dt}=\frac{q(t)}{c_1\gamma_1}. \quad (7)$$

Here, we have introduced the notation

$$J_1=E+c_1T_r,\quad n=c_2\gamma_2\delta_2/c_1\gamma_1\delta_1,$$

$$\theta(t)=\int\limits_0^{\delta(t)}T_1(x,\ t)\,dx, \quad (8)$$

where $\delta(t)=\delta_1-\xi(t)$ is the thickness of the remaining layer of material. Moreover, it is assumed that $\xi(0)=0$. A second equation relating the unknowns $T_2(t)$ and $\xi(t)$ must follow from the boundary conditions.

We note that Eq. (7) is integrated in the finite form

$$\xi(t)=\int\limits_0^t\frac{q(t)}{\gamma_1J_1}\,dt-\frac{c_1}{J_1}\left[n\delta_1(T_2-T_0)+\theta-\theta_0\right]. \quad (9)$$

In accordance with our assumption concerning the thermal conductivities of the rods we will further assume that at the boundary between the layers the condition

$$\text{at } x=\delta(t)$$

$$\frac{\partial T_1}{\partial x}=0 \quad (10)$$

is satisfied. In this case the presence of a second rod is taken into account only in the heat balance equation (9).

As in the case of a semiinfinite rod [1] we take the temperature profile for $0\le x\le\delta(t)$ in the form of a quadratic parabola. We do not risk obtaining a picture similar to that shown in Fig. 2, since by condition (10) we locate the unique minimum of the curve $T_1(x,t)$ (the vertex of the parabola) at the boundary of the region considered $x=\delta(t)$. Thus,

$$T_1(x,\ t)=T_2(t)+(T_r-T_2)\left[1-x/\delta(t)\right]^2. \quad (11)$$

Then $\theta(t)=(T_r+2T_2)\delta/3$, and Eq. (9) becomes

$$\xi(t)=\int\limits_0^t\frac{q(t)}{\gamma_1J_1}\,dt-\frac{c_1}{J_1}\left[n\delta_1(T_2-T_0)+\right.$$

$$\left.+\delta(t)(T_r+2T_2)/3-\delta_1(T_r+2T_0)/3\right]. \quad (12)$$

Equation (12) relates to two unknowns $\xi(t)$ and $T_2(t)$. In order to obtain a second equation in these quantities we employ the condition at the subliming surface (6), whence after substitution of (11) we have

$$T_2(t)=T_r-\frac{\delta}{2\lambda_1}\left(q+\gamma_1E\frac{d\delta}{dt}\right). \quad (13)$$

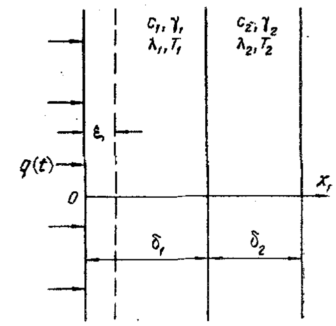

Fig. 1. Schematic representation of the problem.

By means of this relation we eliminate the quantity $T_2(t)$ from (12):

$$\frac{d\delta}{dt}=-\frac{q(t)}{\gamma_1E}-\frac{3\lambda_1}{c_1\gamma_1}\,\frac{1}{\delta+3n\delta_1/2}+\frac{\delta_1\lambda_1}{\gamma_1E}\times$$

$$\times\left((3n-1)T_r-(3n+2)T_0+\right.$$

$$\left.+\frac{3J_1}{c_1}\left(1-\int\limits_0^t\frac{q(t)}{\gamma_1J_1\delta_1}\,dt\right)\right)\times$$

$$\times\left(\delta^2+3n\delta_1\delta/2\right)^{-1}. \quad (14)$$

From Eq. (14) we can easily determine the total sublimation time for the first rod t_{fr}. For this purpose we must set $\delta(t_{fr})=0$; then

$$\int\limits_0^{t_{fr}}\frac{q(t)}{\gamma_1J_1\delta_1}\,dt=$$

$$=1+\frac{c_1}{J_1}\left[\left(n-\frac{1}{3}\right)T_r-\left(n+\frac{2}{3}\right)T_0\right]. \quad (15)$$

We will first consider the case of constant heat flux q = const. We write Eq. (14) in dimensionless form, introducing the notation

$$\bar{\delta}=\bar{\delta}(\bar{t})=\delta(t)/\delta_1;$$

$$\bar{t}=t/t_{fr};\quad p=2\lambda_1\gamma_1J_{eff}^2/3c_1q^2;$$

$$J_{eff}=E+c_1(T_r-T_0);\quad k=E/c_1(T_r-T_0).$$

Some of these quantities were used in [1].

Thus, we obtain

$$\bar{\delta}(\bar{\delta}+b_6)\frac{d\bar{\delta}}{d\bar{t}}=b_1\bar{\delta}(\bar{\delta}+b_6)+$$

$$+b_2\bar{\delta}+b_3(b_4-b_5\bar{t})+b_7, \quad (16)$$

where

$$b_1=-\frac{1}{3}\,\frac{(k+1)^2\bar{t}_{fr}}{k(1+km)};$$

$$b_2 = -\frac{1}{2}\left(\frac{k+1}{1+km}\right)^2 \bar{t}_{fr};$$

$$b_3 = -b_2/3k;\ b_4 = 3k+2;\ b_5 = -3kb_1;$$

$$b_6 = 3n/2;\ b_7 = -nb_2/k;\ \bar{t}_{fr} = t_{fr}/p.$$

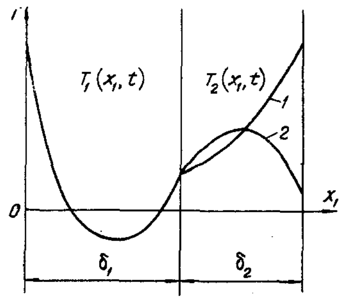

Fig. 2. Possible temperature in a two-layer wall when given by segments of parabolas: 1) thermally insulated surface in the section $x_1 = \delta_1 + \delta_2$; 2) constant temperature at the surface $x_1 = \delta_1 + \delta_2$.

Here, the parameter m characterizes the time τ of heating and sublimation of the first rod up to arrival of the thermal front at the boundary between the layers and, in accordance with [1], is found from the equation

$$\delta(\tau) = (1+km)\delta_0.$$

In deriving Eq. (16) it was assumed that at the moment of arrival of the thermal front at the boundary between the layers the thickness of the heat-assimilating layer [1] and the thickness of the first layer (length of rod exposed to the external heat flux) coincide. The initial condition for Eq. (16) is $\bar{\delta}(0) = 1$. We will find the solution in the form of a power series:

$$\bar{\delta}(\bar{t}) = 1 + \sum_{i=1}^{\infty} c_i \bar{t}^i,\quad 0 \leqslant \bar{t} \leqslant 1. \qquad (17)$$

Substituting (17) in Eq. (16) and equating the coefficients of like powers of the variable $\bar{t}$ on the left and right sides, we obtain

$$c_1 = b_1 + (b_2 + b_3 b_4 + b_7)/(1+b_6);$$

$$c_2 = 0.5\,[c_1(2+b_6)(b_1 - c_1) + b_2 c_1 - b_3 b_5]/(1+b_6);$$

$$c_i = \frac{1}{i(1+b_6)}\left\{b_1\left[(2+b_6)c_{i-1} + \sum_{j=1}^{i-2} c_j c_{i-j-1}\right] + \right.$$

$$\left. + b_2 c_{i-1} - \sum_{j=1}^{i-1} j c_j \left[(2+b_6)c_{i-j} + \sum_{s=1}^{i-j-1} c_s c_{i-j-s}\right]\right\},$$

$$i = 3,\ 4,\ \ldots. \qquad (18)$$

The question of the conditions necessary and sufficient to establish the radius of convergence of series (17) is still not clear. The fact is that the coefficients of the series depend on each other in a complicated way: each successive coefficient is expressed in terms of all the preceding ones.

By direct calculation it is easy to establish that at small values of the parameter n (roughly $n \leqslant 1$) the radius of convergence of series (17) is close to unity. Comparison of the results of calculating $\delta(t)$ from (17) with the data obtained by solving Eq. (16) numerically on a computer indicates that acceptable accuracy can be achieved by keeping the first 3–5 terms of the series.

As n increases, the radius of convergence decreases and at $n = 6.5$ (maximum value of n considered) is approximately 0.4–0.45. Calculations were made for $k = 1.7565$, $m = 1$. From Eq. (15) on going over to dimensionless parameters with $m = 1$ we obtain for t_{fr} the expression

$$\bar{t}_{fr} = (3n + 3k + 2)/(k+1). \qquad (19)$$

The results of calculating $\bar{\delta}(\bar{t})$ for a series of values of the parameter n are presented in Fig. 3a. Moreover, the same figure shows the results of a numerical solution of the starting system of equations on a computer. At small values of n the results of calculating $\bar{\delta}(\bar{t})$ by different methods are in quite good agreement.

Now, solving Eq. (12) for the temperature of the second layer $T_2(t)$ and going over to dimensionless quantities, we have

$$\frac{T_2(t) - T_0}{T_r - T_0} = \frac{1}{2}\,\frac{1}{b_6 + \bar{\delta}(\bar{t})}\,[b_5\bar{t} - b_4(1 - \bar{\delta}(\bar{t}))]. \qquad (20)$$

The temperature distribution in the first layer is determined from expression (11), where the thickness of the layer $\delta(t)$ is calculated from (17).

Thus, for $q = \text{const}$ the problem is completely solved. In Fig. 3b we present the results of a calculation of $(T_2 - T_0)/(T_r - T_0)$ as a function of dimensionless time. The calculations were made both by using the approximate solution obtained above and by integrating the starting system of equations (1) and (2) by the method of finite differences on a computer. It is clear that the approximate calculation of the temperature of the second layer $T_2(t)$ correctly reflects the qualitative picture of the heating process. Relatively large deviations from the numerical solution are observed on the initial interval. This is attributable to the fact that at the outset the approximate representation of the heating process in the form of a finite heat-assimilating layer has a considerable effect. This effect is more strongly manifested at small values of n.

Since using the solution of Eq. (16) in form (17) is rather complicated owing to the clumsiness of expressions (18) for the series coefficients, it is worthwhile to consider the following approximation of $\bar{\delta} = \bar{\delta}(\bar{t})$:

$$\bar{\delta}(\bar{t}) = (1 - \bar{t})^{\beta}, \qquad (21)$$

where

$$\beta = \dot{\xi}\, t_{fr}/\delta_1.$$

Taking the quantity $\bar{\delta}(\bar{t})$ in this form ensures that the initial and final values of the unknown function coincide with $\bar{\delta}(0) = 1$, $\bar{\delta}(1) = 0$ and the initial value of the rate of displacement of the sublimation front with $d\xi/dt =$

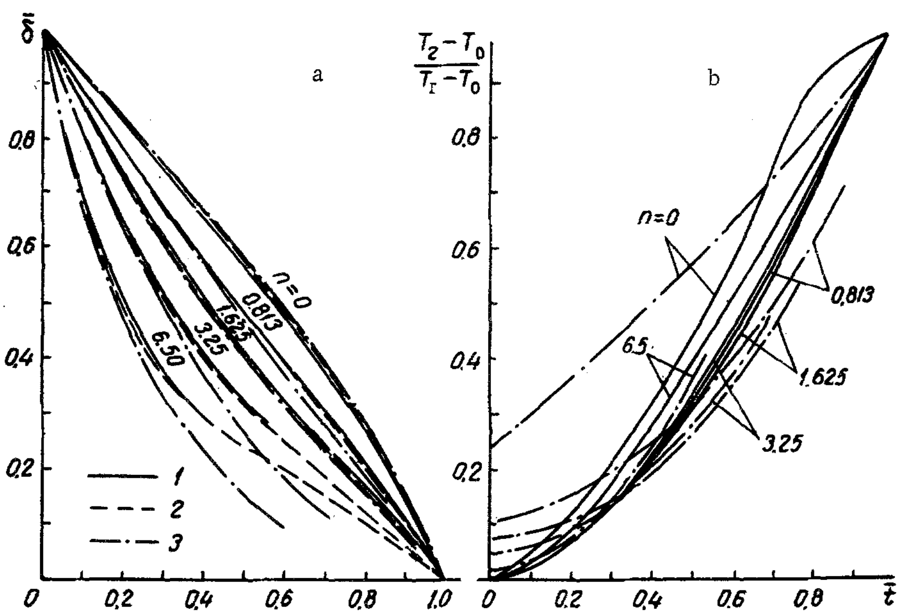

Fig. 3. Comparison of the results of an approximate calculation of the values of $\bar{\delta}(\bar{t})$ and $(T_2 - T_0)/(T_\Gamma - T_0)$ at q = const and the data of a numerical solution: 1) approximate solution; 2) numerical solution of Eq. (16); 3) numerical solution of the starting system of equations.

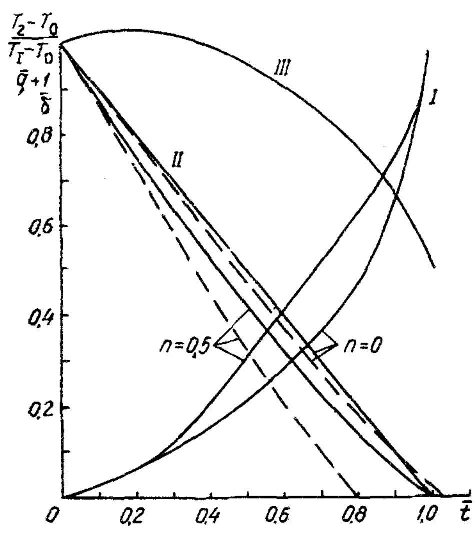

Fig. 4. Thickness of first layer and temperature of second layer as a function of time for a variable heat flux: solid lines—approximate solution; dashed lines—numerical solution for a semiinfinite rod: I) for $(T_2 - T_0)/(T_\Gamma - T_0)$, II) for $\bar{\delta}$, III) for $\bar{q} + 1$.

$= \dot{\xi}_0$, and also gives satisfaction of the condition $d\bar{t} <$ < 0 which follows from the physical picture of the process. Using the expression obtained in [1] for $\dot{\xi} = \dot{\xi}(m)$ and going over to dimensionless parameters, it is easy to show that

$$\beta = \bar{t}_{fr} \frac{k+1}{3} \frac{2km+m-k}{(1+km)^2}.$$

At m = 1 we obtain $\beta = \bar{t}_{fr}/3$. By a direct simple calculation it can be established that expression (21) quite accurately approximates the real relation at small values of the parameter n. It may be assumed that this holds roughly up to n = 1.5–2. At larger n the proposed formula gives only a qualitative picture of the event.

We will now consider the case of a variable heat flux at the boundary q = q(t). We represent this function in the form

$$q(t) = q_0 + q_1(t), \tag{22}$$

where the subscripts denote the heat flow to the surface of the body at the moment of arrival of the thermal front at the boundary between the two media (t = 0) and the subsequent deviations of the heat flux from the indicated value.

We shall find the solution of Eq. (14) for q(t) from (22) in the form of a sum

$$\delta(t) = \delta_1(t) + \delta_2(t).$$

The first term on the right side of this equation is the solution of Eq. (14) at constant heat flux $q = q_0$, the second, the deviation of the solution from the steady-state value for an arbitrary heat flux.

We assume that $|\delta_2(t)| \ll \delta_1(t)$. Then, expanding the expressions

$$\frac{1}{1+\delta_2(t)/\delta_1(t)} \text{ and } \frac{1}{1+\delta_2(t)/(\delta_1(t)+b_6)}$$

in series as infinitely decreasing progressions and restricting ourselves to terms containing the denominators of the progressions in powers not higher than the first, after going over to dimensionless parameters we have the equation

$$df(\bar{t})/d\bar{t} = -S_1(\bar{t})f(\bar{t}) + S_2(\bar{t}), \tag{23}$$

where

$$S_1(\bar{t}) = (2\bar{\delta}+b_6)\frac{b_3(b_4-b_5\bar{t})+b_7}{\bar{\delta}^2(\bar{\delta}+b_6)^2} +$$

$$+\frac{b_2}{(\bar{\delta}+b_6)^2} - b_3b_5\frac{2\bar{\delta}+b_6}{\bar{\delta}^2(\bar{\delta}+b_6)^2}\int_0^{\bar{t}} \bar{q}\,d\bar{t};$$

$$S_2(t) = b_1\bar{q} - \frac{b_3b_5}{\bar{\delta}(\bar{\delta}+b_6)}\int_0^{\bar{t}} \bar{q}\,d\bar{t};$$

$$\bar{q} = q_1(t)/q_0;\ \bar{\delta} = \delta_1(t)/\delta_1,\quad f(\bar{t}) = \delta_2(t)/\delta_1.$$

As before, the total sublimation time of the layer δ_1 is given by Eq. (15), and instead of q = const we everywhere introduce the quantity q_0.

By definition $f(0) = 0$. Then the solution of Eq. (23) is the function

$$f(\bar{t}) = \exp\left[-\int_0^{\bar{t}} S_1(\bar{t})\,d\bar{t}\right]\int_0^{\bar{t}} S_2(\bar{t}) \times$$

$$\times \exp\left[\int_0^{\bar{t}} S_1(\bar{t})\,d\bar{t}\right]d\bar{t}. \tag{24}$$

Now, using the relation between ξ(t) and δ(t), it is easy to determine the time dependence of the sublimed layer and, using expression (12), find the temperature of the second layer (rod):

$$\frac{T_2(t)-T_0}{T_r-T_0} =$$

$$= \frac{1}{2}\frac{1}{b_6+\bar{\delta}(\bar{t})}\left[b_5\int_0^{\bar{t}}(1+\bar{q})\,d\bar{t} - b_4(1-\bar{\delta}(\bar{t}))\right]. \tag{25}$$

The temperature distribution along the length of the first rod is approximately described by relation (11).

Thus, the problem of the sublimation and heating of a two-layer rod (or plate) with an arbitrary heat flow at the boundary has been approximately solved. The solution is presented in the form of quadratures (24), (25).

It should be noted that under certain conditions complete sublimation of the first rod may not be achieved. This may happen if the heat flow to the body falls to zero before the rod in question has evaporated. In this case instead of t_{fr} it is necessary to use another characteristic, for example, the parameter p. This does not require any significant changes in the solution.

Figure 4 presents the results of a calculation of the sublimation and heating of a two-layer system (rods or plates) for a variable heat flow at the boundary. The calculations were made for k = 1.7565, n = 0 and n = = 0.5. In this case the quantity $\delta_1(t)$ was calculated using approximation (21). For comparison the same figure presents the results of a calculation of the thickness of the subliming layer for a semiinfinite rod obtained by numerical solution of the corresponding system of equations [1] on a computer. It is clear that at n ~ 0 the layer of material sublimes more rapidly in the case of a rod of finite length. On the other hand, even at n ~ 0.5 the effect of heat transfer to the second layer is so great that total sublimation of the first rod occurs much later than observed for the same amount of sublimation of a semiinfinite rod.

The proposed approximate method of solving the problem of heating and sublimation of a two-layer plate (rod) may prove useful for estimating purposes.

NOTATION

c is the specific heat of the material; γ is the density of material; λ is the thermal conductivity; *a* is the thermal diffusivity of the material; E is the latent heat of sublimation of the material; δ is the thickness of the layer of material (length of rod); t is the time; x_1 is the coordinate directed from outside surface into the interior of the wall; $\xi(t)$ is the coordinate of the sublimation front; $\dot{\xi}(t) = d\xi/dt$ is the velocity of the sublimation front; $T(x_1, t)$ is the temperature; $q(t)$ is the amount of heat supplied to the unit surface area in a unit time. The subscripts 1, 2 denote that the characteristics correspond to the first and second layers (rods), respectively.

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